THE TRAJECTORIES OF DYNAMICS*

BY

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The object of this paper is to investigate the geometric character of the trajectories described by a particle moving freely in a plane under the action of any force which depends only upon the position of the particle.† The equations of motion are thus of the form ‡

$$m\frac{d^2x}{dt^2} = \phi(x, y), \qquad m\frac{d^2y}{dt^2} = \psi(x, y).$$

At the initial time, say $t = 0$, the particle may be projected from any position $x = x_0$, $y = y_0$, with any velocity, given, for instance, by the initial slope $y' = y'_0$ and the initial speed $v = v_0$. A unique trajectory is then generated. By varying the arbitrary constants x_0, y_0, y'_0, v_0, we obtain in all, since each trajectory may be described from any one of its points, ∞^3 trajectories. Thus each field of force gives rise to a definite triply infinite system of curves.

The properties of such systems which are obtained in this paper are entirely general, that is, they hold for all (positional) forces. That such properties must exist is seen most readily from the fact that the differential equation of the third order which represents a dynamical system is of special form (article 1). If a triply infinite system of curves is selected arbitrarily, its differential equation will not in general be of this form, so there will exist no field of force having the given curves for trajectories. (A single trajectory has, of course, no peculiar properties.)

A first set of geometric properties is obtained by considering the ∞^1 trajectories obtained by starting particles at a given point in a given direction with all possible speeds (articles 3–11). . If for each of these curves the osculating parabola is constructed at the given point, the locus of the foci of the parabolas so obtained is a circle passing through the given point. If the point is kept

* Presented to the Society April 29, 1905. Received for publication April 10, 1906.

† The theory for several particles, and for motion on a surface, in space, etc., will be discussed in later papers.

‡ The functions ϕ and ψ are assumed to be uniform and to possess first and second derivatives in the region of the plane considered. The case where the force vanishes everywhere is excluded; the trajectories are then straight lines and thus constitute only a doubly infinite system.

fixed and the initial direction varied, the corresponding circle varies in a definite manner. In particular, the locus of the centers of the ∞^1 circles obtained is a conic about the given point as focus; or, what is equivalent, the envelope of the circles is itself a circle.

It is found that these properties belong to other systems of curves besides dynamical trajectories, and all such systems are explicitly determined (articles 12–16). The narrowest class of systems thus obtained involves four arbitrary functions of x, y, and is thus far more general than the dynamical class since the latter involves only two, namely, ϕ and ψ.

The next set of properties derived depends on the consideration of hyperosculating circles, i. e., circles of curvature which have contact of the third order with a trajectory (articles 17–23). The first two properties obtained hold for the more general systems referred to above. Essentially distinct properties are then obtained, first in analytic, next in geometric form.

The whole set of properties now obtained is shown to be *completely characteristic* of systems of dynamical trajectories (art. 24). The properties are sufficient as well as necessary. Any triply infinite system of curves having the properties in question can be identified with the trajectories produced by a certain (positional) force. This force is uniquely determined except for a constant factor.

Articles 9 and 10 relate to two special cases of interest. The first deals with conservative forces; here the locus, referred to above, which is in general a conic having the given point as focus, degenerates into a straight line (counted twice). The second deals with the type of force described by LECORNU as "analytic," which is of interest from the mathematical rather than from the physical point of view. The conic here becomes a circle. In both cases the properties are characteristic (art. 25).

In the last part of the paper it is shown that collineations are the only point transformations which convert every system of trajectories into such a system. The same projective group also leaves the more general types of systems previously obtained invariant (articles 26–29). The result is finally applied to remove an essential limitation from APPELL's discussion of the transformation theory of dynamical problems.

THE DIFFERENTIAL EQUATION OF THE TRAJECTORIES.

1. Without loss of generality, we may assume that the particle is of unit mass; so that the equations defining its motion are

$$\ddot{x} = \phi(x, y), \qquad \ddot{y} = \psi(x, y), \tag{1}$$

where dots denote derivatives with respect to the time t.

To obtain the differential equation of the trajectories it is necessary to eliminate the time. Along any trajectory y is a function of x, and, if accents are used to denote differentiation with respect to x, we have the following relations between the *geometric* and *kinematic* derivatives:

$$(2) \qquad y' = \frac{\dot{y}}{\dot{x}}, \qquad y'' = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^3}.$$

$$(3) \qquad y''' = \frac{\dot{x}(\dot{x}\dddot{y} - \dot{y}\dddot{x}) - 3\ddot{x}(\dot{x}\ddot{y} - \dot{y}\ddot{x})}{\dot{x}^5}.$$

Two additional equations containing third derivatives are obtained by differentiating equations (1) with respect to t. These are

$$(4) \qquad \dddot{x} = \phi_x\dot{x} + \phi_y\dot{y}, \qquad \dddot{y} = \psi_x\dot{x} + \psi_y\dot{y},$$

where the subscripts denote partial derivatives.

We now have seven equations from which all the time derivatives $\dot{x}$, $\dot{y}$, $\ddot{x}$, $\ddot{y}$, $\dddot{x}$, $\dddot{y}$ may be eliminated. The result is

$$(5) \qquad (\psi - y'\phi)y''' = \{\psi_x + (\psi_y - \phi_x)y' - \phi_y y'^2\}y'' - 3\phi y''^2.$$

This is the differential equation of the ∞^3 *trajectories corresponding to the general field of force* (1).

We introduce, for abbreviation,

$$(5') \qquad P = \frac{\psi_x + (\psi_y - \phi_x)y' - \phi_y y'^2}{\psi - y'\phi}, \qquad Q = \frac{-3\phi}{\psi - y'\phi},$$

so that (5) may be written

$$(6) \qquad y''' = Py'' + Qy''^2.$$

2. If a triply infinite system of plane curves is selected arbitrarily, its equation will not, in general, be of the special form (5) and hence no force exists of which the given curves are the trajectories. If on the other hand the equation is of form (5), it follows that there is at least one corresponding field (1). Can two fields have the same system of trajectories?

If in one field the components are ϕ, ψ and, in the other, ϕ_1, ψ_1, then, by assumption, the corresponding equations (5) must coincide; that is

$$(7) \qquad P = P_1, \qquad Q = Q_1.$$

From the second equation there results

$$\frac{\psi_1}{\phi_1} = \frac{\psi}{\phi},$$

so that we may put

$$\phi_1 = \lambda\phi, \qquad \psi_1 = \lambda\psi,$$

where λ is some function of x, y.

Introducing these values in the first equation (7), we find that λ_x and λ_y both vanish, so that λ is simply a constant κ. Therefore $\phi_1 = \kappa\phi$, $\psi_1 = \kappa\psi$.

If two fields of force produce the same system of trajectories, the forces are the same except for a constant factor. The constant factor corresponds merely to a change in the unit of force. Thus the system of trajectories belonging to a given field (1) *completely defines that field.*

Osculating parabolas.

3. Just as a set of values for x, y, y', y'', that is, a differential element of second order, is pictured most simply by means of the corresponding circle of curvature, so a differential element of the third order, defined by x, y, y', y'', y''', may be pictured by the unique osculating parabola. We collect here the general formulas to be used in the subsequent discussion.

If the given element is assumed to be at the origin, so that its coördinates are $(0, 0, y', y'', y''')$, the corresponding parabola is

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0, \tag{8}$$

where

$$\begin{aligned} &a = (3y''^2 - y'y''')^2, \qquad b = y'''(3y''^2 - y'y'''), \qquad c = y'''^2, \\ &d = 9y'y''^2, \qquad e = -9y''^3, \qquad f = 0. \end{aligned} \tag{9}$$

The coördinates α, β of the focus of any parabola (8), where of course $b^2 - ac = 0$, are given by the formulas

$$\begin{aligned} 2\alpha &= \frac{1}{(be - cd)(a + c)}\{c(d^2 - e^2 + fc - fa) - 2b(de - bf)\}, \\ 2\beta &= \frac{1}{(be - cd)(a + c)}\{b(d^2 - e^2 + fc - fa) + 2c(de - bf)\}, \end{aligned} \tag{10}$$

Substituting the values (9), we have

$$\begin{aligned} 2\alpha &= \frac{-3y''\{y'''(y'^2 - 1) + 2y'(3y''^2 - y'y''')\}}{y'''^2 + (3y''^2 - y'y''')^2}, \\ 2\beta &= \frac{-3y''\{(3y''^2 - y'y''')(y'^2 - 1) - 2y'y'''\}}{y'''^2 + (3y''^2 - y'y''')^2}. \end{aligned} \tag{11}$$

4. Consider now any triply infinite system of curves, defined by a differential equation of third order

$$(12) \qquad y''' = f(x, y, y', y'').$$

Through a given point and in a given direction there pass ∞^1 curves of the system. Each of these has a definite osculating parabola at the given point. The locus of the foci of these parabolas we shall term the *focal curve*. Thus to each lineal element (x, y, y') there corresponds a definite focal curve.*

The form of the focal curve depends, of course, upon the form of the differential equation. Since x, y, y' are fixed, y''' is a certain function of y''. Substituting this in (11), α and β are expressed in terms of y''. The elimination of y'' gives then the finite equation of the required locus.

5. We proceed now to apply these considerations to systems of dynamical trajectories. The equation of such a system is of the particular form

$$(6) \qquad y''' = Py'' + Qy''^2.$$

There is no difficulty in calculating the focal curves. Here P and Q have fixed values since they involve only x, y, y'. Substituting in (11) and carrying out the elimination, we find

$$(13) \quad 2P(\alpha^2+\beta^2) + \{3(y'^2-1) - y'(y'^2+1)Q\}\alpha + \{(y'^2+1)Q - 6y'\}\beta = 0.$$

Here the current coördinates α, β are referred to the fixed point (x, y) as origin. Hence each focal curve is a circle passing through the given point.

THEOREM I. *If in any field of force* (1) *a particle is projected from a given point in a given direction, to each initial speed there corresponds a definite trajectory. The ∞^1 trajectories obtained by varying the initial speed are so situated that the locus of the foci of the osculating parabolas constructed at the given point is a circle passing through that point.*

Thus each field of force gives rise to a definite correspondence between the lineal elements (x, y, y') on the one hand and the circles on the other. The explicit equation of the circle corresponding to (x, y, y') is †

$$(14) \quad \begin{aligned} &2\{\psi_x + (\psi_y - \phi_x)y' - \phi_y y'^2\}\{(\alpha-x)^2 + (\beta-y)^2\} \\ &\quad + 3\{\psi y'^2 + 2\phi y' - \psi\}(\alpha - x) + 3\{\phi y'^2 - 2\psi y' - \phi\}(\beta - y) = 0. \end{aligned}$$

6. Assuming the fixed lineal element to be $x = 0$, $y = 0$, $y' = 0$, the circle (14) reduces to

$$(15) \qquad 2\psi_x(\alpha^2+\beta^2) - 3\psi\alpha - 3\phi\beta = 0.$$

* The totality of focal curves for all lineal elements is, in general, a triply infinite system It may, however, in certain cases be of smaller dimensionality. To each focal system corresponds an infinitude of original systems.

† Here the α, β coördinates refer to the same axes as the x, y coördinates.

At the same time the differential equation (5) gives, for the ∞^1 trajectories defined by the given element,

$$(16)\qquad y''' = \frac{\psi_x}{\psi} y'' - \frac{3\phi}{\psi} y''^2 .$$

As the initial speed is changed, y'' changes, and therefore also the radius of curvature ρ, which in the present case is

$$\rho = \frac{1}{y''} .$$

The locus of centers of curvature is of course simply the normal, here the axis of y.

The focus of the osculating parabolas is simultaneously describing the circle (15), its coördinates being

$$\alpha = \frac{3y''y'''}{2(y'''^2 + 9y''^4)}, \qquad \beta = \frac{9y''^2}{2(y'''^2 + 9y''^4)} .$$

If now we let m denote the slope of the line connecting the focus with the origin, we find

$$m = \frac{3\psi y''}{\psi_x - 3\phi y''} .$$

The introduction of ρ gives

$$(17)\qquad m = \frac{3\psi}{\psi_x \rho - 3\phi} \qquad \text{or} \qquad \rho = \frac{3\phi m + 3\psi}{\psi_x m} .$$

This is a bilinear relation between m and ρ. Therefore

*As the initial speed varies (the initial position and direction being fixed), the center of curvature of the trajectory describes a straight line (the normal) and the focus of the osculating parabola describes a circle, in such a way that the two ranges (one linear, the other circular) are homographically related.**

7. From the equation (15) we see that the circle passes through the origin in a direction whose slope is $-\psi/\phi$. But ψ/ϕ is the slope of the vector representing the force acting at the given point. These two directions are thus symmetrically situated with respect to the axis of x, which is the assumed initial direction of the trajectories.

THEOREM II. *The circle that corresponds, according to theorem I, to a given lineal element (x, y, y') is so situated that the element bisects the angle between the force acting at the given point and the tangent to the circle at that point.*

* Furthermore, the given point, which is situated on both ranges, corresponds to itself.

8. We now consider the ∞^1 circles corresponding to the ∞^1 lineal elements at a given point. We may take the given point as the origin, but the slope y' is necessarily a parameter.

Denoting by X, Y the coördinates of the center of any one of these circles, we have, from (14),

$$(18) \quad X = -\frac{3\{\psi y'^2 + 2\phi y' - \psi\}}{4\{\psi_x + (\psi_y - \phi_x)y' - \phi_y y'^2\}}, \quad Y = -\frac{3\{\phi y'^2 - 2\psi y' - \phi\}}{4\{\psi_x + (\psi_y - \phi_x)y' - \phi_y y'^2\}}.$$

As y' varies, the point X, Y describes the curve

$$(19) \quad \begin{aligned} &(\psi_x - \phi_y)\sqrt{\phi^2 + \psi^2}\sqrt{X^2 + Y^2} + (\psi_x + \phi_y)(\psi X + \phi Y) \\ &\qquad + (\phi_x - \psi_y)(\phi X - \psi Y) - \tfrac{3}{2}(\phi^2 + \psi^2) = 0. \end{aligned}$$

This is a conic with its focus at the origin. Our result is then

THEOREM III. *The locus of the centers of the ∞^1 circles associated, according to theorem I, with the ∞^1 lineal elements at a given point, is a conic with one focus at that point.*

Theorems II and III contain a complete discription of the arrangement of the ∞^1 circles considered.

It is easy to deduce from the result just obtained the form of the envelope of the circles. This is simply the director circle of the conic (19).

THEOREM III′. *The envelope of the ∞^1 circles considered is itself a circle.*

9. *Conservative force.* These results simplify considerably in two important special cases which we now consider. In the first place the conic (19) degenerates into a straight line (counted twice) if

$$(20) \quad \phi_y - \psi_x = 0.$$

This is precisely the case in which the force (1) is *conservative*; that is, when there exists a work function or potential W such that

$$(20') \quad \phi = W_x, \qquad \psi = W_y.$$

*When the given force is conservative, the locus of the centers of the circles considered in theorem III is, for every point of the plane, a straight line. This property is characteristic of the conservative case.**

10. *Analytic force.* In the other special case the conic (19) reduces to a circle. This is so when

$$(21) \quad \phi_x - \psi_y = 0, \qquad \phi_y + \psi_x = 0.$$

* When the force is not conservative (20) represents a curve, and only for the points of this curve is the corresponding conic a straight line.

These are the conditions under which ϕ, ψ are conjugate harmonic functions, i. e., under which $\phi + i\psi$ is a function of the complex variable $x + iy$. Such forces have been investigated by LECORNU,† who describes them as *analytic forces.*

When, and only when, the force is "analytic," in the sense of LECORNU, *is the conic of theorem III, for every point of the plane, a circle.*

The enveloping circle of theorem III′ then has its center at the given point. It follows that the ∞^1 focal circles corresponding to the lineal elements at any point all have the same radius.

11. In the general case there corresponds, by theorem III, a definite conic (19) to each point x, y of the plane. One focus is at the given point and the other is

$$
(21)\qquad
\begin{aligned}
\bar{x} &= x + \frac{3\{\psi(\phi_y + \psi_x) + \phi(\phi_x - \psi_y)\}}{(\phi_y - \psi_x)^2 - \{(\phi_x - \psi_y)^2 + (\phi_y + \psi_x)^2\}},\\
\bar{y} &= y + \frac{3\{\phi(\phi_y + \psi_x) - \psi(\phi_x - \psi_y)\}}{(\phi_y - \psi_x)^2 - \{(\phi_x - \psi_y)^2 + (\phi_y + \psi_x)^2\}}.
\end{aligned}
$$

Thus each field of force gives rise to a definite correspondence between the points of the plane.

In the "analytic" case every point corresponds to itself. All points $\bar{x}$, $\bar{y}$ are at infinity when

$$(\phi_x - \psi_y)^2 + (\phi_y + \psi_x)^2 - (\phi_y - \psi_x)^2 = 0,$$

that is, when,

$$(22)\qquad (\phi_x - \psi_y)^2 + 4\phi_y\psi_x = 0.$$

In this special case the conic (19) is a parabola and the envelope considered in theorem III′ is a straight line.

CONVERSE QUESTIONS.

12. We proceed now to inquire to what extent the geometric properties obtained for general dynamical systems of trajectories are characteristic of such systems. If a triply infinite system of plane curves has one or more of these properties does it follow that it corresponds to a field of force (1)? It will be shown that this is not the case. Thus certain classes of systems will be brought to light which include the dynamical class as a special case.

13. Our first problem is to convert theorem I. Find all triply infinite systems possessing the property that for every lineal element the corresponding focal curve is a circle passing through the point of the element. We shall refer to this as *property* I.

† *Sur les forces analytiques*, Journal de l'Ecole Polytecnique, vol. 55 (1885), pp. 253–274.

The only systems of curves possessing property I are those defined by a differential equation of the form

$$(23) \qquad y''' = G(x, y, y')y'' + H(x, y, y')y''^2,$$

where G and H are arbitrary functions of x, y, y'.

By assumption, the foci α, β of the osculating parabolas of the ∞^1 curves defined by any lineal element x, y, y' lie on a circle,

$$A_0(\alpha^2 + \beta^2) + A_1\alpha + A_2\beta = 0,$$

through the point x, y which is taken as the origin for the α, β coördinates. Here the coefficients may involve x, y, y' in any way. Substituting the general values of α, β given in (11), we find, by direct reduction, that the differential equation is of type (23).

This type is much more general than the dynamical type (5) or (6), since in the latter P and Q are special functions of x, y, y'.

The circle corresponding to a lineal element is, for (23), given by

$$(24) \qquad 2G(\alpha^2 + \beta^2) + \{3(y'^2 - 1) - y'(y'^2 + 1)H\}\alpha + \{(y'^2 + 1)H - 6y'\}\beta = 0.$$

It is easily shown that all systems (23) have the homographic property considered in article 6.*

14. In order to make it possible to convert theorem II, it is of course necessary to state the property there proved in a form which does not assume the existence of a field of force. We state it as

Property II. There exists for each point (x, y) of the plane a certain direction such that the angle between this direction and the tangent to the focal circle corresponding to any element (x, y, y') at the given point, is bisected by that element.

The particular direction, of course, may vary from point to point. Its slope is thus an arbitrary function ω of x, y.

The problem is to find those systems (23) which have property II. The slope of the circle (24) at the given point is

$$\frac{3(y'^2 - 1) - y'(y'^2 + 1)H}{6y' - (y'^2 + 1)H}.$$

* If no restriction is placed on the focal locus, but it is required that the line joining the given point O to the focus (α, β) shall describe a pencil homographic to the range described by the center of curvature, then the differential equation belongs to the more general type

$$y''' = \frac{F_1 + F_2 y''}{F_3 + F_4 y''} y''^2,$$

where the F's are any functions of x, y, y'. The focal locus is then found to be a quartic curve.

The bisection property is translated by the condition

$$\frac{\omega - y'}{1 + \omega y'} = \frac{y'\{6y' + (1 + y'^2)H\} - 3(y'^2 - 1) + y'(y'^2 + 1)}{6y' - (y'^2 + 1)H + y'\{3(y'^2 - 1) - y'(y'^2 + 1)H\}}.$$

Solving this for H, we find

$$H = \frac{3}{y' - \omega}. \tag{25}$$

The only systems possessing properties I *and* II *are those of the type* (23) *for which H is of the special form* (25). *The differential equation is then*

$$(y' - \omega)y''' = (y' - \omega)Gy'' + 3y''^2, \tag{26}$$

where G is any function of x, y, y' and ω is any function of x, y.

This class is obviously more general than the dynamical class (5). In dynamical systems the fixed direction whose slope is ω is, of course, the direction of the force acting at the given point.

15. The question suggested by theorem III is that of finding systems which, in addition to property I, have

Property III. The locus of the centers of the ∞^1 circles corresponding to the elements at a given point is a conic with that point as a focus.

For the equation (23) the focal circles are given by (24). The center X, Y is then

$$X = \frac{y'(y'^2 + 1)H - 3(y'^2 - 1)}{4G}, \qquad Y = \frac{6y' - (y'^2 + 1)H}{4G}. \tag{27}$$

The equation of a conic with its focus at the given point, which is the origin of the X, Y coördinates, is

$$B_0\sqrt{X^2 + Y^2} + B_1X + B_2Y + B_3 = 0. \tag{28}$$

The coefficients may involve x, y, y' in any way.

From (27), we have

$$\sqrt{X^2 + Y^2} = \frac{(y'^2 + 1)\sqrt{(y'^2 + 1)H^2 - 6y'H + 9}}{4G}. \tag{27$'$}$$

Substituting in (28), we find

$$\begin{aligned} B_0(y'^2 + 1)\sqrt{(y'^2 + 1)H^2 - 6y'H + 9} + B_1\{y'(y'^2 + 1)H - 3(y'^2 - 1)\} \\ + B_2\{6y' - (y'^2 + 1)H\} + 4B_3G = 0. \end{aligned} \tag{28$'$}$$

Solving this for G, and changing the notation slightly, we express our result as follows:

The only systems possessing properties I and III are those defined by an equation of type (23) in which

$$G = \lambda_1(1+y'^2)\sqrt{(1+y'^2)H^2 - 6y'H + 9} \quad (29)$$
$$+ \lambda_2\{y'(y'^2+1)H - 3(y'^2-1)\} + \lambda_3\{6y' - (y'^2+1)H\}.$$

The differential equation thus involves one arbitrary function H of x, y, y and three arbitrary functions $\lambda_1, \lambda_2, \lambda_3$ of x, y.

16. We now inquire for the systems possessing all three of the properties considered.

Putting the results obtained in the last three paragraphs together, namely, combining formulas (23), (25), and (29), and changing the notation slightly, we find that

The only systems of curves possessing properties I, II, and III are those whose differential equation is of the form

$$(y' - \omega)y''' = \{\lambda y'^2 + \mu y' + \nu\}y'' + 3y''^2, \quad (30)$$

where ω, λ, μ, ν are arbitrary functions of x, y.

The class (30) obviously includes the dynamical type (5) as a special case.* The geometric properties obtained up to this point hold for the total class (30) and hence are not by themselves sufficient to characterize dynamical systems.† We therefore proceed to obtain additional geometric properties.

Hyperosculating circles.

17. Under what conditions will the circle of curvature constructed for a trajectory at a given point have contact of the third order with the trajectory?

The differential equation of all circles is

$$(1+y'^2)y''' = 3y'y''^2. \quad (31)$$

The differential equation of the trajectories is

$$y''' = Py'' + Qy''^2. \quad (6)$$

In case of hyperosculation the values of y''' found from these equations must be the same. Hence ‡

* The comparison shows that the derivatives y', y'', y''' are involved in exactly the same form. The distinction is with respect to the generality of the functions of x, y appearing as coefficients.

† It is, however, easy to show that if any ∞^2 curves passing through a point are given, such that properties I, II and III hold for that point, it is possible to find (in infinitely many ways) a field of force such that each of the ∞^2 trajectories passing through the given point has contact of the third order with one of the given curves.

‡ We disregard the factor y'' leading to straight lines; these enter as limiting cases in every system of trajectories.

$$P + Qy'' = \frac{3y'y''}{1 + y'^2}.$$

Solving for y'', and substituting from (5′), we find

$$(32) \qquad y'' = \frac{\{\psi_x + (\psi_y - \phi_x)y' - \phi_y y'^2\}(1 + y'^2)}{3(\phi + y'\psi)}.$$

This equation shows that to any set of values of x, y, y' there corresponds one value of y''. Hence to each lineal element there corresponds one trajectory which is hyperosculated by its circle of curvature.

The center of curvature for any element of the second order, referred to the given point as origin, is

$$(33) \qquad X = -\frac{y'(1 + y'^2)}{y''}, \qquad Y = \frac{1 + y'^2}{y''}.$$

Inversely,

$$(33') \qquad y' = -\frac{X}{Y}, \qquad y'' = \frac{X^2 + Y^2}{Y^3}.$$

The result of substituting these values in (32) is

$$(35) \qquad \phi_y X^2 - (\phi_x - \psi_y)XY - \psi_x Y^2 + 3(\phi Y - \psi X) = 0.$$

This represents a conic passing through the given point.

THEOREM IV. *In each direction through a given point there passes one trajectory which has contact of the third order with its circle of curvature. The locus of the centers of the* ∞^1 *hyperosculating circles, obtained by varying the initial direction, is a conic passing through the given point.*

The equation of the tangent to the conic (19) at the origin is

$$(35') \qquad \phi Y - \psi X = 0.$$

The slope of this line is ψ/ϕ. Hence

THEOREM IV_1. *The conic of theorem IV passes through the given point in the direction of the force acting at that point.*

18. In the special cases considered in articles 9 and 10 we find, by means of (35), these results:

When the force (1) *is conservative, the conic associated by theorem IV with each point of the plane is an equilateral hyperbola. Conversely, if for every point the conic is an equilateral hyperbola, the force is conservative.*

The conic is a circle for each point when, and only when, the force is "analytic."

We note that the condition that (35) is a parabola leads to the same equation (22) that was obtained in article 11. Hence

The conic of theorem IV is a parabola when, and only when, this is true for the conic of theorem III.

19. We now consider the converse questions connected in the general results obtained in article 17. The property considered in theorem IV will be referred to as *property IV*. *Property IV_1*, involved in theorem IV_1, is to be stated in a form which does not assume the existence of a field of force; this may be done by replacing the direction of the force by the special direction considered in theorem II.

It is easily seen that the discussions of article 17 apply to all equations of the class (30). The locus of centers is in fact given by the equation

$$\lambda X^2 - \mu XY + \nu Y^2 + 3(Y - \omega X) = 0, \tag{36}$$

which takes the place of (35). The tangent at the origin is

$$Y - \omega X = 0. \tag{36'}$$

It follows that

Every triply infinite system of plane curves which has the properties I, II, and III also has the properties IV and IV_1.

It is thus seen that theorems IV and IV_1, relating to hyperosculating circles, are consequences of the theorems previously obtained relating to osculating parabolas. A direct derivation may be obtained by noting that a circle which has contact of third order with a trajectory must also have contact of third order with the osculating parabola of the trajectory. This is possible only when the point of contact is the vertex of the parabola. Then the radius of the circle is half the latus rectum of the parabola, and the focus is situated on the normal at the given point.

It is thus seen that the focal circle and the hyperosculating circle which correspond uniquely to any given lineal element are intimately connected. If the former is given the latter may be constructed as follows. At the given point O, erect a perpendicular to the lineal element meeting the focal circle again at a point D. Prolong this line to C so that $DC = OD$. Then the hyperosculating circle has the center C and the radius CO. If only the point O and the focal circle are given, the corresponding lineal element is obtained, according to theorem II, by bisecting the angle between the tangent to the circle and the vector representing the force acting at O. The ∞^1 focal circles corresponding to all elements at O are so situated, by theorem III, that locus of their centers is a conic with O as focus. If for each of these circles the point C is constructed as explained above, it is shown without difficulty that the locus so obtained is a conic passing through O in the direction of the force acting at O. This proves theorems IV and IV_1.

20. We now inquire for all systems which have property IV together with some of the previous properties. The discussions are omitted, only results being stated.

If a system of curves has properties I *and* IV, *its equation is of form* (23) *with* G *and* H *connected by the relation*

$$(37) \qquad (\delta_0 y'^2 + \delta_1 y' + \delta_2)\{(1 + y'^2)H - 3y'\} = (\delta_3 + y'\delta_4)G,$$

where the δ*'s are arbitrary functions of* x, y.

Properties I, III, *and* IV *give rise to a class of equations, defined by the relations* (29) *and* (37), *involving eight arbitrary functions of* x, y.

Properties I, II *and* IV *define the equations*

$$(38) \qquad (y' - \omega)y''' = \frac{(\lambda y'^2 + \mu y' + \nu)(1 + \omega y')}{(\gamma_1 y' + \gamma_2)} y'' + 3y''^2$$

containing five essential coefficients.

The conic considered in the fourth property is then

$$(38') \qquad \lambda X^2 - \mu XY + \nu Y^2 - 3(\gamma_1 X - \gamma_2 Y) = 0.$$

The tangent at the origin is

$$\gamma_1 X - \gamma_2 Y = 0.$$

If the slope of this is ω, (38) reduces to (30). Hence

The systems of type (30) *are completely characterized by properties* I, II, IV *and* IV_1.

Thus theorem III is a consequence of theorems I, II, IV, and IV_1. This may also be shown from the geometric considerations given in the preceding article.

Complete characterization.

21. It remains now to examine just how the dynamical systems, defined by equations of type (5), are to be distinguished in the general class (30).

If an equation of type (30) is of the special type (5), we have, by comparison,

$$(39) \qquad \omega = \frac{\psi}{\phi},$$

$$(40) \qquad \lambda = \frac{\phi_y}{\phi}, \qquad \mu = \frac{\phi_x - \psi_y}{\phi}, \qquad \nu = -\frac{\psi_x}{\phi}.$$

Substituting

$$(39') \qquad \psi = \omega\phi,$$

and introducing, for convenience,

$$(41) \qquad \bar{\phi} = \log \phi \qquad \text{or} \qquad \phi = e^{\bar{\phi}}$$

we find, from (40),

(42) $$\lambda = \bar{\phi}_y, \qquad \mu = \bar{\phi}_x - \omega\bar{\phi}_y - \omega_y, \qquad \nu = -\,\omega\bar{\phi}_x - \omega_x.$$

From the first and third of this set,

(43) $$\bar{\phi}_x = -\frac{\nu + \omega_x}{\omega}, \qquad \bar{\phi}_y = \lambda.$$

Substituting in the second of the set,

(44) $$\lambda\omega^2 + \mu\omega + \nu + \omega_x + \omega\omega_y = 0.$$

It furthermore follows from (43) that

(45) $$\lambda_x + \left(\frac{\nu + \omega_x}{\omega}\right)_y = 0.$$

In the dynamical case the functions λ, μ, ν, ω thus satisfy the relations (44) and (45). We now prove that these relations are sufficient.

In virtue of (45) it is possible to find a function $\bar{\phi}$ so as to satisfy both equations (43). On account of (44) the equations (42) then necessarily hold. Finally, (41) and (39′) determine a pair of functions ϕ, ψ, that is, a field of force. The integration constant connected with the determination of $\bar{\phi}$ appears merely as a constant factor in ϕ and ψ, as it should in accordance with the result obtained in article 2.

An equation of the form (30) *represents a system of dynamical trajectories when, and only when, the functions* λ, μ, ν, ω, *which appear as coefficients, satisfy the relations* (44) *and* (45).

22. The geometric interpretation of (44) is quite simple. To each point O, there corresponds, in virtue of property IV, a definite conic (36) passing through O. The slope of the tangent at O is ω. The equation of the normal is

$$X + \omega Y = 0.$$

This intersects the conic at some point N. The distance intercepted is found to be

(46) $$ON = \frac{3(1 + \omega^2)^{\frac{3}{2}}}{\lambda\omega^2 + \mu\omega + \nu}.$$

Consider now the curves defined by the differential equation

(47) $$y' = \omega(x, y).$$

(In the dynamical case these are the lines of force). The curve (46) which passes through O is tangent to the conic (36). Its radius of curvature,

$$\rho = \frac{(1 + y'^2)^{\frac{3}{2}}}{y''},$$

is given by

(48) $$\rho = \frac{(1 + \omega^2)^{\frac{3}{2}}}{\omega_x + \omega\omega_y}.$$

By means of (46) and (48), the relation (44) is observed to be equivalent to

$$\frac{1}{\rho} \pm \frac{3}{ON} = 0.$$

A discussion shows that N is on the same side of O as the center of curvature. Hence the lower sign is here appropriate, so that

(49) $$ON = 3\rho.$$

We thus find that all systems (30) in which the coefficients fulfill the relation (44) possess the following property, which we refer to as

Property V. The normal at any point O to the conic associated with that point according to property IV, intersects the conic again at a distance which is three times the radius of curvature of the curve (47) passing through O. The latter curve is defined geometrically by the fact that its direction at any point is the direction considered in property II, or, what is the same, the direction of the conic corresponding to that point by IV.

Every system of trajectories fulfills the relation (44) in question. Observing that ON is the radius of a certain hyperosculating circle and that (47) represents the lines of force, we may state the result as follows:

THEOREM V. *Of the trajectories which pass through a given point in the direction of the force acting at that point, there is one which has contact of the third order with its circle of curvature; the radius of curvature of this trajectory is three times the radius of curvature of the line of force passing through the given point.*

23. The interpretation of the second relation, (45), is not so simple since here the derivatives of the functions λ and ν are involved.

The intercepts of the conic (36) on the axes of coördinates are found to be

(50) $$OA = \frac{3\omega}{\lambda}, \qquad OB = -\frac{3}{\nu}.$$

Hence λ and ν have the following geometric interpretations:

(50′) $$\lambda = \frac{3\omega}{OA}, \qquad \nu = -\frac{3}{OB}.$$

The function ω, we have seen, represents the slope of the conic at the given point. Thus all the quantities appearing in (45) have definite geometric significance. The relation is

$$\frac{\partial}{\partial x}\left(\frac{\omega}{OA}\right)-\frac{\partial}{\partial y}\left(\frac{1}{\omega\cdot OB}\right)+\frac{1}{3}\frac{\partial}{\partial y}\left(\frac{\omega_x}{\omega}\right)=0. \tag{51}$$

This may be expressed in more symmetrical form as follows. At A draw a line parallel to the axis of ordinates intersecting the normal ON in some point

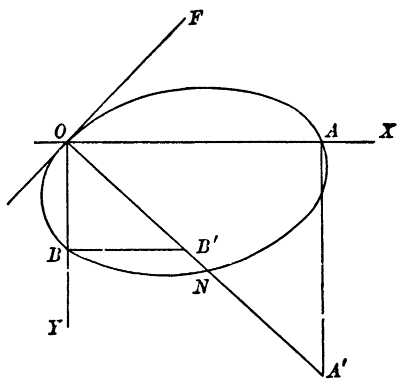

FIG. 1.

A'. Similarly, at B draw a line parallel to the axis of abscissas intersecting the normal in B'. Then

$$AA'=\frac{OA}{\omega},\qquad BB'=-\omega\cdot OB.$$

Introducing these quantities in (51), we have the following result:

Any system (30) in which the coefficients satisfy the relation (45) has, in addition to properties I, II, IV, IV_1, the *property* VI expressed by

$$\frac{\partial}{\partial x}\frac{1}{AA'}+\frac{\partial}{\partial y}\frac{1}{BB'}+\frac{\omega\omega_{xy}-\omega_x\omega_y}{3\omega^2}=0. \tag{52}$$

In particular,

THEOREM VI. *Every system of dynamical trajectories possesses property* VI.

24. Every dynamical system has the properties I, II, III (or III′), IV, IV_1, V, VI. Conversely, since the equations (30) are characterized by either I, II, III, or I, II, IV, IV_1, it follows, from the result stated at the close of article 21, that dynamical systems may be characterized by either

(Σ_1) I, II, III, V, VI,

or else

(Σ_2) I, II, IV, IV_1, V, VI.

If a triply infinite system of plane curves has the set of properties (Σ_1), *or the set* (Σ_2), *it is possible to find a field of force* (1) *so that, for any initial conditions, a particle moving in the field describes one of the given curves.*

In the set (Σ_2), IV_1 may be conveniently combined with II. The modified set is

(Σ_3) IV, I, II′, V, VI,

where property II′ differs from property II by requiring the direction which is fixed at each point to be the direction of the tangent to the conic corresponding to that point according to IV.

The properties entering in each of the characteristic sets Σ_1, Σ_2, Σ_3 are all necessary in the sense that none of the properties can be derived from those which precede it in the set. The sets may then be described as *ordinally* independent. (The sets are in all probability *absolutely* independent, but the author has not attempted the requisite discussion.)

With reference to VI, it is sufficient to require this property for any one set of rectangular axes. It then necessarily holds for all rectangular axes. Hence it is really a geometric property of the system.

25. From the results obtained in articles 9 and 10, it follows that

Triply infinite systems of curves capable of generation as the trajectories of a CONSERVATIVE *force are completely characterized by the properties* (Σ_1) *together with the requirement that the conic involved in* III *shall be a straight line; or by* (Σ_3) *with the requirement that the conic of* IV *shall be an equilateral hyperbola.*

Systems which can be generated by an "ANALYTIC" *force are characterized by* (Σ_1) *with the conic of* III *required to be a circle; or by* (Σ_3) *with the same requirement imposed on the conic of* IV.

Transformations

26. We consider in the following articles the effect of an arbitrary point transformation upon systems of trajectories.

Let the point transformation be

$$X = \Phi(x, y), \qquad Y = \Psi(x, y). \tag{53}$$

The first extended transformation we write in the form

$$(cy' + d)Y' = ay' + b, \tag{53'}$$

where

$$a = \psi_y, \qquad b = \psi_x, \qquad c = \phi_y, \qquad d = \phi_x.$$

The second extension is

(53″) $$(cy' + d)^3 Y'' = \alpha + \beta y' + \gamma y'^2 + \delta y'^3 + jy'',$$

where j is the jacobian

$$j = ad - bc,$$

and

$$\alpha = \Phi_x \Psi_{xx} - \Psi_x \Phi_{xx},$$

$$\beta = \Phi_y \Psi_{xx} - \Psi_y \Phi_{xx} + 2\Phi_x \Psi_{xy} - 2\Psi_x \Phi_{xy},$$

$$\gamma = \Phi_x \Psi_{yy} - \Psi_x \Phi_{yy} + 2\Phi_y \Psi_{xy} - 2\Psi_y \Phi_{xy},$$

$$\delta = \Phi_y \Psi_{yy} - \Psi_y \Phi_{yy}.$$

Finally, the third extension is

(53‴) $$\begin{aligned}(cy' + d)^5 Y''' = (cy' + d)(A_0 + A_1 y' + A_2 y'^2 + A_3 y'^3 + A_4 y'^4) \\ - 3(\alpha + \beta y' + \gamma y'^2 + \delta y'^3)(\alpha_2 + \beta_2 y' + \gamma_2 y'^2) \\ + (\alpha_1 + \beta_1 y' + \gamma_1 y'^2) y'' - 3cjy''^2 + j(cy' + d) y''',\end{aligned}$$

where

$$\alpha_1 = d(\beta + j_x) - 3c\alpha - 3jd_x,$$

$$\beta_1 = c(\beta + j_x) + d(2\gamma + j_y) - 3c\beta - 3j(c_x + d_y),$$

$$\gamma_1 = c(2\gamma + j_y) + 3d\delta - 3c\gamma - 3jc_y;$$

$$\alpha_2 = \Phi_{xx}, \qquad \beta_2 = 2\Phi_{xy}, \qquad \gamma_2 = \Phi_{yy};$$

$$A_0 = \alpha_x,\ A_1 = \alpha_y + \beta_x,\ A_2 = \beta_y + \gamma_x,\ A_3 = \gamma_y + \delta_x,\ A_4 = \delta_y.$$

It will be convenient to write (55″) and (53‴) in more abbreviated fashion as follows:

(56) $$Y'' = \epsilon_0 + \epsilon_1 y'', \qquad Y''' = E_0 + E_1 y'' + E_2 y''^2 + E_2 y'''.$$

The coefficients here involve y' as well as x and y.

27. From (56) it is observed that an arbitrary transformation applied to an equation of type

(23) $$y''' = G(xyy')y'' + H(xyy')y''^2$$

converts it into one of the form

(57) $$y''' = f_0(xyy') + f_1(xyy')y'' + f_2(xyy')y''^2$$

For what transformations will the type (23) be invariant? In this case f_0 must vanish. This gives the condition

(58) $$\bar{G}\epsilon_0 + \bar{H}\epsilon_0^2 = E_0,$$

where the bars indicate the result of applying the transformation to the functions G and H.

Since this condition is to be fulfilled for all values of G and H, we may, in particular, assume them to be arbitrary constants. Then $\bar{G}$ and $\bar{H}$ are arbitrary constants, so that ϵ_0 and E_0 must vanish. The first formula in (56) then shows that the required transformations leave invariant the differential equation $y'' = 0$. The only transformations with this property are the collineations.

Conversely, if the transformation is a collineation, the coefficients ϵ_0 and E_0 really vanish; the extended transformations are, in fact, of the form

(59) $$\begin{aligned} (cy' + d)Y' &= ay' + b, \\ (cy' + d)^3 Y'' &= jy'', \\ (cy' + d)^5 Y''' &= (\alpha_1 + \beta_1 y' + \gamma_1 y'^2)y'' - 3cjy''^2 + j(cy' + d)y'''. \end{aligned}$$

Thus a collineation converts (23) into one of the same type

$$Y''' = G_1 Y'' + H_1 Y''^2,$$

the new coefficients being related to the old as follows:

(60) $$G_1 = \frac{j(cy' + d)^2 \bar{G} - (\alpha_1 + \beta_1 y' + \gamma_1 y'^2)}{j(cy' + d)^2}, \qquad H_1 = \frac{j\bar{H} + 3c(cy' + d)}{(cy' + d)^2}.$$

The only point transformations which convert all equations of type (23) *into equations of the same type are the projective transformations.*

Recalling the fact that a system of curves whose differential equation is of type (23) is characterized by the possession of property I, we may restate the result as follows:

Property I is invariant under collineations, but not invariant under any other point transformation.

Thus, if any number of curves which pass through a common point in a common direction are so situated that the foci of their osculating parabolas (at the given point) are located on a circle passing through the given point, the same is true after the curves are projected in any way.*

28. We now prove that *properties II and III are invariant under projective transformation.*

For a system possessing the former property (in addition, of course, to property I), the coefficient H is of the form

(61) $$H = \frac{3}{y' - \omega(x, y)}.$$

---

* If an arbitrary point transformation is applied, the new focal locus is found to be a quartic curve of special type.

From (53′), we have

$$\overline{H} = \frac{3(cy' + d)}{ay' + b - \omega(cy' + d)}.$$

Introducing this value in the second formula of (60), we find

$$(61') \qquad H_1 = \frac{3(a - c\omega)}{ay' + b - \omega(cy' + d)} = \frac{3}{y' - \dfrac{d\omega - b}{a - c\omega}},$$

which is of the form characteristic of property II.

In a similar fashion, it may be shown that, if G is of the form

$$(62) \qquad G = \frac{\lambda y'^2 + \mu y' + \nu}{y' - \omega},$$

the same is true of the coefficient G_1 in the transformed differential equation.

The coefficients (61) and (62) are precisely those involved in the type (30). Therefore

Any system of curves whose equation is of type (30), *that is, any system possessing properties* I, II *and* III, *is converted by projective transformation into a system of the same type. For no other transformations is this the case.*

29. We consider finally systems of dynamical trajectories, the differential equation then being of type (5).

If a point transformation is to convert every system of this kind into a system of the same kind, it is necessary that

$$(63) \qquad \overline{P}\epsilon_0 + \overline{Q}\epsilon_0^2 - E_0 = 0.$$

Here $\overline{P}$, $\overline{Q}$ denote the results obtained by applying the transformation considered to the coefficients P, Q defined by (5′). This equation is to hold for all dynamical systems, i. e., for all functions ϕ and ψ.

Take first the particular force

$$\phi = 0, \qquad \psi = 1.$$

The coefficients P, Q then vanish; hence also the quantities $\overline{P}$, $\overline{Q}$. Therefore, from (63), E_0 must vanish, and the condition reduces to

$$(63') \qquad \overline{P}\epsilon_0 + \overline{Q}\epsilon_0^2 = 0.$$

Take, in the next place, the particular force

$$\phi = 1, \qquad \psi = 0.$$

Then

$$P = 0, \qquad Q = \frac{3}{y'}, \qquad \overline{P} = 0, \qquad \overline{Q} = 3\,\frac{cy' + d}{ay' + d}.$$

The last quantity cannot vanish. For, if it did, the function Φ, of which c and d are the partial derivatives, would reduce to a constant, so that the point transformation (53) would be degenerate. It follows therefore, from (63′), that ϵ_0 must vanish. Hence the transformation is necessarily a collineation. It is easily verified that all collineations have the required property. Hence

The only point transformations which convert every system of dynamical trajectories into such a system are the collineations of the plane. Thus the group for which the type (5) *is invariant is the eight-parameter projective group.*

TRANSFORMATIONS INVOLVING THE TIME.

30. The importance of projective transformations in dynamics was first indicated by APPELL.* His discussion is based on the equations

$$\frac{d^2x}{dt^2} = \phi(x, y), \qquad \frac{d^2y}{dt^2} = \psi(x, y). \tag{64}$$

It is shown that for any collineation

$$x_1 = \frac{ax + by + c}{a''x + b''y + c''}, \qquad y_1 = \frac{a'x + b'y + c'}{a''x + b''y + c''}, \tag{65}$$

it is possible to find a transformation of the time, namely,

$$t_1 + \text{const.} = \frac{t}{k(a''x + b''y + c'')^2}, \tag{65'}$$

so that the equations (64), written in the new variables, are of the same form

$$\frac{d^2x_1}{dt_1^2} = \phi_1(x_1, y_1), \qquad \frac{d^2y_1}{dt_1^2} = \psi_1(x_1, y_1). \tag{66}$$

The corresponding transformation of the force components is

$$\begin{aligned} \phi_1 &= k^2(a''x + b''y + c)^2\{C'(x\psi - y\phi) + B'\phi - A'\psi\}, \\ \psi_1 &= k^2(a''x + b''y + c'')^2\{-C(x\psi - y\phi) - B\phi + A\psi\}, \end{aligned} \tag{67}$$

where the capitals denote minors in the determinant of the linear transformation (65).

The converse is discussed in a limited form: it is proved that no transformations of the type

$$x_1 = \Phi(x, y), \qquad y_1 = \Psi(x, y), \qquad t_1 + \text{const} = t\lambda(x, y), \tag{68}$$

* *De l'homographie en mécanique*, American Journal of Mathematics, vol. 12 (1890), pp. 103–114. An application to BERTRAND's problem is given in vol. 13, pp. 153–158 of the same journal. HALPHEN had applied a particular homography in an earlier paper.

except those given by (65), (65′), can convert every system (64) into a like system.

The result holds, however, even if no special assumption is made concerning the transformation of t. We prove namely this stronger converse:

The only transformations of the type

$$x_1 = \Phi(x, y), \qquad y_1 = \Psi(x, y), \qquad t_1 = \chi(x, y, t), \tag{69}$$

for which every system (64) *is converted into a system of the same form, are the Appell transformations represented by* (65) *with* (65′).

To show this, we apply the result proved in article 29. The point transformation is necessarily a collineation, so that (69) must be of the form

$$x_1 = \frac{ax + by + c}{a''x + b''y + c''}, \qquad y_1 = \frac{ax + by + c}{a'x + b'y + c'}, \qquad t_1 = \chi(x, y, t). \tag{70}$$

Let this transformation be denoted by S' and let the Appell transformation (65), (65′) be denoted by S.

Since both transformations leave the type (64) invariant, the same is true of their combination $S'S^{-1}$, which is of the form

$$x_1 = x, \qquad y_1 = y, \qquad t_1 = f(x, y, t). \tag{71}$$

We now determine the function f. The new time derivatives are related to the old as follows:

$$\dot{x}_1 = \frac{\dot{x}}{f_x\dot{x} + f_y\dot{y} + f_t}, \qquad \ddot{x}_1 = \frac{(f_x\dot{x} + f_y\dot{y} + f_t)\ddot{x} - \dot{x}F}{(f_x\dot{x} + f_y\dot{y} + f_t)^3},$$

$$\dot{y}_1 = \frac{\dot{y}}{f_x\dot{x} + f_y\dot{y} + f_t}, \qquad \ddot{y}_1 = \frac{(f_x\dot{x} + f_y\dot{y} + f_t)\ddot{y} - \dot{y}F}{(f_x\dot{x} + f_y\dot{y} + f_t)^3},$$

where

$$F = f_x\ddot{x} + f_y\ddot{y} + f_{xx}\dot{x}^2 + 2f_{xy}\dot{x}\dot{y} + f_{yy}\dot{y}^2 + f_{xt}\dot{x} + f_{yt}\dot{y} + f_{tt}.$$

From these formulas it is found that (71) converts (64) into

$$\begin{aligned} (f_x\dot{x} + f_y\dot{y} + f_t)\ddot{x} - \dot{x}F &= \phi(x, y)(f_x\dot{x} + f_y\dot{y} + f_t)^3, \\ (f_x\dot{x} + f_y\dot{y} + f_t)\ddot{y} - \dot{y}F &= \psi(x, y)(f_x\dot{x} + f_y\dot{y} + f_t)^3. \end{aligned} \tag{72}$$

By hypothesis these are to reduce to a system

$$\ddot{x} = \phi_1(x, y), \qquad \ddot{y} = \psi_1(x, y).$$

Substituting these values in (72), multiplying the first by $\dot{y}$, the second by $\dot{x}$, and subtracting, we find

$$(f_x\dot{x} + f_y\dot{y} + f_t)^2(\dot{y}\phi - \dot{x}\psi) = \dot{y}\phi_1 - \dot{x}\psi_1. \tag{73}$$

This is to be an identity, so that we may equate like powers of $\dot{x}, \dot{y}$. It is thus easily shown that f_x and f_y vanish, while f_t is simply a constant. Hence f is of the form

$$f = \text{const } t + \text{const}. \tag{74}$$

Having determined the transformation (71), we need simply combine this with S in order to obtain S'. The result is of the APPELL type, as stated in our theorem.

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